

## Buchert coarse-graining and the classical energy conditions

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So-called “Buchert averaging” is actually a coarse-graining procedure, where fine detail is “smeared out” due to limited spatio-temporal resolution. For technical reasons, (to be explained herein), “averaging” is not really an appropriate term, and I shall consistently describe the process as a “coarse-graining”. Because Einstein gravity is nonlinear the coarse-grained Einstein tensor is typically not equal to the Einstein tensor of the coarse-grained spacetime geometry. The discrepancy can be viewed as an “effective” stress-energy. To keep otherwise messy technical issues firmly under control, I shall work with conformal-FLRW (CFLRW) cosmologies. These CFLRW-based models are particularly tractable, and are also particularly attractive observationally: the CMB is not distorted. In this CFLRW context one can prove some rigorous theorems regarding the interplay between Buchert coarse-graining, tracelessness of the effective stress-energy, and the classical energy conditions.

*Keywords:* Buchert averaging; coarse-graining; smearing; FLRW cosmology; CFLRW cosmology.

### 1. Introduction

The cosmological back-reaction problem continues to generate considerable (and sometimes quite heated) debate.<sup>1–11</sup> There is significant disagreement as to just how one should split the universe into “smooth background” plus “local perturbations”, and yet more disagreement as to whether or not these perturbations remain small. I shall work within the framework of conformally-FLRW (CFLRW) cosmologies,<sup>13</sup> where the mathematics and physics are both firmly under control.<sup>13</sup> These CFLRW cosmologies are simply generic FLRW spacetimes distorted by a (possibly non-perturbatively large) conformal factor. Cosmographically<sup>12</sup> this is the unique non-perturbative distortion that can be applied to FLRW cosmologies without grossly modifying the CMB.<sup>13</sup> In counterpoint, when working in this CFLRW framework, Buchert’s coarse-graining procedure simplifies tremendously. The contribution to the effective stress-energy due to coarse graining can be explicitly calculated. This coarse-graining contribution to the effective stress-energy need not be traceless, though it will typically satisfy the classical energy conditions. (Further details are in preparation.<sup>14</sup>)

### 2. Strategy

Let  $g_{ab}$  represent a FLRW spacetime. Adopt conformal time for the FLRW. Then:

$$g_{ab} dx^a dx^b = a(\eta)^2 \left\{ -d\eta^2 + \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right\} = a(\eta)^2 \{ \hat{g}_{ab} dx^a dx^b \}. \quad (1)$$

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Here  $\hat{g}_{ab}$  is translation invariant in both space and (conformal) time. On this FLRW space-time, define coarse-graining by

$$\langle \phi(x) \rangle = \oint f(x-y) \phi(y) d^4y; \quad \oint f(x-y) d^4y = 1. \quad (2)$$

Here  $f(x-y)$  is any non-negative translation-invariant normalized kernel in both space and (conformal) time. The high symmetries of FLRW are essential to this construction.

### 3. Preliminaries

In this FLRW framework, coarse-graining the gradient of a scalar is the same as taking the gradient of the coarse-grained scalar:

$$\langle \nabla_x \phi(x) \rangle = \oint f(x-y) \nabla_y \phi(y) d^4y \quad (3)$$

$$= - \oint [\nabla_y f(x-y)] \phi(y) d^4y \quad (4)$$

$$= + \oint [\nabla_x f(x-y)] \phi(y) d^4y \quad (5)$$

$$= \nabla_x \oint f(x-y) \phi(y) d^4y \quad (6)$$

$$= \nabla_x \langle \phi(x) \rangle. \quad (7)$$

Translation invariance of the kernel is crucial to establishing this result.

Now calculate:

$$\langle \langle \phi(x) \rangle \rangle = \oint f(x-y) \langle \phi(y) \rangle d^4y \quad (8)$$

$$= \oint f(x-y) \left( \oint f(y-z) \phi(z) d^4z \right) d^4y \quad (9)$$

$$= \oint \left( \oint f(x-y) f(y-z) d^4y \right) \phi(z) d^4z \quad (10)$$

$$= \oint \left( \oint f(x-z) f(z-y) d^4z \right) \phi(y) d^4y \quad (11)$$

$$= \oint [f^{\otimes 2}](x-y) \phi(y) d^4y \quad (12)$$

$$\neq \langle \phi(x) \rangle \quad (13)$$

So you have to be very careful. Since  $\langle \langle \phi(x) \rangle \rangle \neq \langle \phi(x) \rangle$ , then if you insist on using the word “averaging” to describe the coarse-graining process, it follows that “the average of the average is not the average”. This is at gross variance with the normal use of the word “averaging”, and for this reason I shall eschew this term. Furthermore:

$$\left\langle \left( \phi(x) - \langle \phi(x) \rangle \right)^2 \right\rangle \neq \langle \phi(x)^2 \rangle - \langle \phi(x) \rangle^2. \quad (14)$$

More precisely

$$\begin{aligned} \langle (\phi(x) - \langle \phi(x) \rangle)^2 \rangle &= \langle \phi(x)^2 \rangle - \langle \phi(x) \rangle^2 \\ &+ \langle \phi(x)^2 \rangle - 2\langle \phi(x) \rangle \langle \phi(x) \rangle + \langle \phi(x) \rangle^2. \end{aligned} \quad (15)$$

For ensemble averages, or indeed any situation where the normal rules of averaging apply, that second line vanishes. But it does not vanish for a coarse-graining process. For these reasons the phrase “coarse-graining” is more appropriate than the word “averaging”. While one cannot now naively assert  $\langle \phi(x)^2 \rangle - \langle \phi(x) \rangle^2 \geq 0$  based on the usual “averaging” arguments, this inequality can instead be derived from positivity of the coarse-graining kernel plus an appeal to the Cauchy–Schwarz inequality:

$$\langle \phi(x) \rangle = \int f(x-y)\phi(y)dy \leq \sqrt{\int f(x-y)\phi(y)^2dy} \sqrt{\int f(x-y)dy} = \sqrt{\langle \phi(x)^2 \rangle}. \quad (16)$$

#### 4. Conformally related spacetimes

Consider first a conformal deformation of a given metric:  $e^{2\phi} g_{ab}$ . We first note a number of utterly standard results.

- Ricci tensor:

$$R(e^{2\phi}g)_{ab} = R(g)_{ab} - 2[\nabla_a \nabla_b \phi - \nabla_a \phi \nabla_b \phi] - g_{ab}[\nabla^2 \phi + 2|\nabla \phi|^2]. \quad (17)$$

- Ricci scalar:

$$g^{ab}R(e^{2\phi}g)_{ab} = g^{ab}R(g)_{ab} - 6[\nabla^2 \phi + |\nabla \phi|^2]. \quad (18)$$

- Einstein tensor:

$$G(e^{2\phi}g)_{ab} = G(g)_{ab} - 2[\nabla_a \nabla_b \phi - \nabla_a \phi \nabla_b \phi] + g_{ab}[2\nabla^2 \phi + |\nabla \phi|^2]. \quad (19)$$

Now apply (generic) coarse-graining.

- Coarse-grain the Einstein tensor:

$$\langle G(e^{2\phi}g)_{ab} \rangle = G(g)_{ab} - 2\langle \nabla_a \nabla_b \phi - \nabla_a \phi \nabla_b \phi \rangle + g_{ab}[\langle 2\nabla^2 \phi + |\nabla \phi|^2 \rangle]. \quad (20)$$

- Coarse-grain the conformal mode:

$$G(e^{2\langle \phi \rangle}g)_{ab} = G(g)_{ab} - 2\nabla_a \nabla_b \langle \phi \rangle - \nabla_a \langle \phi \rangle \nabla_b \langle \phi \rangle + g_{ab}[2\nabla^2 \langle \phi \rangle + |\nabla \langle \phi \rangle|^2]. \quad (21)$$

- Coarse-grain the conformally deformed metric holding the base metric fixed:

$$\begin{aligned} G(\langle e^{2\phi} \rangle g)_{ab} &= G(\langle e^{2\phi} \rangle g)_{ab} = G(g)_{ab} - \left[ \nabla_a \nabla_b \ln \langle e^{2\phi} \rangle - \frac{1}{2} \nabla_a \ln \langle e^{2\phi} \rangle \nabla_b \ln \langle e^{2\phi} \rangle \right] \\ &+ g_{ab} \left[ \nabla^2 \ln \langle e^{2\phi} \rangle + \frac{1}{4} |\nabla \ln \langle e^{2\phi} \rangle|^2 \right]. \end{aligned} \quad (22)$$

This so far holds for any generic coarse-graining process, details as yet unspecified. (We have not even used any assumed translation invariance for the kernel in the coarse-graining process.)

## 5. CFLRW spacetimes

For CFLRW space-time smearing this simplifies considerably.

- Coarse-grain the Einstein tensor:

$$\langle G(e^{2\phi}g)_{ab} \rangle = G(g)_{ab} - 2\nabla_a \nabla_b \langle \phi \rangle - \langle \nabla_a \phi \nabla_b \phi \rangle + g_{ab}[2\nabla^2 \langle \phi \rangle + \langle |\nabla \phi|^2 \rangle]. \quad (23)$$

- Coarse-grain the conformal mode:

$$G(e^{2\langle \phi \rangle}g)_{ab} = G(g)_{ab} - 2\nabla_a \nabla_b \langle \phi \rangle - \nabla_a \langle \phi \rangle \nabla_b \langle \phi \rangle + g_{ab}[2\nabla^2 \langle \phi \rangle + |\nabla \langle \phi \rangle|^2]. \quad (24)$$

- Coarse-grain the metric:

$$\begin{aligned} G(\langle e^{2\phi}g \rangle)_{ab} = G(\langle e^{2\phi} \rangle g)_{ab} = G(g)_{ab} - \left[ \nabla_a \nabla_b \ln \langle e^{2\phi} \rangle - \frac{1}{2} \nabla_a \ln \langle e^{2\phi} \rangle \nabla_b \ln \langle e^{2\phi} \rangle \right] \\ + g_{ab} \left[ \nabla^2 \ln \langle e^{2\phi} \rangle + \frac{1}{4} |\nabla \ln \langle e^{2\phi} \rangle|^2 \right]. \end{aligned} \quad (25)$$

Combining the above, for CFLRW coarse-graining we have:

$$\langle G(e^{2\phi}g)_{ab} \rangle = G(e^{2\langle \phi \rangle}g)_{ab} - 2[\langle \nabla_a \phi \nabla_b \phi \rangle - \nabla_a \langle \phi \rangle \nabla_b \langle \phi \rangle] + g_{ab}[\langle |\nabla \phi|^2 \rangle - |\nabla \langle \phi \rangle|^2]. \quad (26)$$

This is a specific, (very high symmetry), case of Buchert coarse-graining, one where all the technical details are now fully under control.

## 6. Effective stress-energy

The coarse-grained spacetime “sees” the effective stress-energy

$$T_{ab}^{\text{effective}} = \langle T_{ab} \rangle + \Delta T_{ab}^{\text{Buchert}}. \quad (27)$$

The part of the “effective stress tensor” due to CFLRW coarse-graining is:

$$\Delta T_{ab}^{\text{Buchert}} = 2[\langle \nabla_a \phi \nabla_b \phi \rangle - \nabla_a \langle \phi \rangle \nabla_b \langle \phi \rangle] - g_{ab}[\langle |\nabla \phi|^2 \rangle - |\nabla \langle \phi \rangle|^2]. \quad (28)$$

Note that this vanishes (as it should) in the limit where the smearing kernel becomes a delta function (so that coarse-graining is switched off). For the trace of this Buchert contribution to the effective stress tensor we have

$$\Delta T^{\text{Buchert}} = -2 \{ \langle |\nabla \phi|^2 \rangle - |\nabla \langle \phi \rangle|^2 \} = 2 \{ \langle \dot{\phi}^2 \rangle - \nabla \dot{\phi}^2 \} - 2 \sum_{i=1}^3 \{ \langle \nabla_i \phi^2 \rangle - \nabla_i \langle \phi \rangle^2 \}. \quad (29)$$

This need not be zero, and generically is in fact nonzero. Since the first term is positive and the last three terms are negative, the trace can easily be arranged to be either positive or negative. In contrast, this effective stress energy typically will satisfy many of the other classical energy conditions.

— For instance, for null vectors, using the FLRW symmetries to assert  $k^a \nabla_a \langle \phi \rangle = k^a \langle \nabla_a \phi \rangle = \langle k^a \nabla_a \phi \rangle$ , we have:<sup>15–20</sup>

$$\Delta T_{ab}^{\text{Buchert}} k^a k^b = 2 \{ \langle (k^a \nabla_a \phi)^2 \rangle - \langle k^a \nabla_a \phi \rangle^2 \} \geq 0. \quad (30)$$

So the null energy condition (NEC) is satisfied for this form of coarse-graining.

— For timelike vectors, we have:<sup>15–20</sup>

$$\Delta T_{ab}^{\text{Buchert}} V^a V^b = \{g^{ab} + 2V^a V^b\} \{ \langle \nabla_a \phi \nabla_b \phi \rangle - \nabla_a \langle \phi \rangle \nabla_b \langle \phi \rangle \}. \quad (31)$$

Use a tetrad decomposition with  $e_0^a = V^a$  being the timelike vector  $V^a$  of interest, then

$$g^{ab} + 2V^a V^b = V^a V^b + \sum_{A=1}^3 e_A^a e_A^b = \sum_{A=0}^3 e_A^a e_A^b. \quad (32)$$

This is a positive definite Euclidean signature “metric”. Now using the FLRW symmetries to assert  $e_A^a \nabla_a \langle \phi \rangle = e_A^a \langle \nabla_a \phi \rangle = \langle e_A^a \nabla_a \phi \rangle = \langle \nabla_{e_A} \phi \rangle$ , we have

$$\Delta T_{ab}^{\text{Buchert}} V^a V^b = \sum_{A=0}^3 \{ \langle (e_A^a \nabla_a \phi)^2 \rangle - \langle e_A^a \nabla_a \phi \rangle^2 \} \geq 0. \quad (33)$$

So the weak energy condition (WEC) is satisfied for this form of coarse-graining.

— For the strong energy condition (SEC) consider the trace-reversed quantity<sup>15–20</sup>

$$\overline{\Delta T_{ab}^{\text{Buchert}}} = \Delta T_{ab}^{\text{Buchert}} - \frac{1}{2} g_{ab} (\Delta T_{cd}^{\text{Buchert}} g^{cd}) \quad (34)$$

$$= 2[ \langle \nabla_a \phi \nabla_b \phi \rangle - \nabla_a \langle \phi \rangle \nabla_b \langle \phi \rangle ]. \quad (35)$$

Then for timelike vectors

$$\overline{\Delta T_{ab}^{\text{Buchert}}} V^a V^b = 2 \{ \langle \nabla_a \phi \nabla_b \phi \rangle - \nabla_a \langle \phi \rangle \nabla_b \langle \phi \rangle \} V^a V^b \quad (36)$$

$$= 2 \{ \langle (\nabla_V \phi)^2 \rangle - \langle \nabla_V \phi \rangle^2 \} \geq 0. \quad (37)$$

So the strong energy condition (SEC) is satisfied for this form of coarse-graining.

## 7. Cosmological setting — $w$ parameter

In a cosmological setting, on large enough regions, one might (approximately) hope:

$$\langle \nabla_i \phi \nabla_j \phi \rangle = \langle (\phi')^2 \rangle g_{ij}; \quad \langle \dot{\phi} \nabla_i \phi \rangle = 0; \quad \langle \nabla_i \phi \rangle = 0. \quad (38)$$

Then

$$\langle \nabla_a \phi \nabla_b \phi \rangle \rightarrow \left[ \begin{array}{c|c} \langle (\dot{\phi})^2 \rangle & 0 \\ \hline 0 & \langle (\phi')^2 \rangle g_{ij} \end{array} \right], \quad (39)$$

implying

$$\Delta T_{ab}^{\text{Buchert}} \rightarrow \left[ \begin{array}{c|c} \langle (\dot{\phi})^2 \rangle + 3\langle (\phi')^2 \rangle - \langle \dot{\phi} \rangle^2 & 0 \\ \hline 0 & (\langle (\dot{\phi})^2 \rangle - \langle (\phi')^2 \rangle - \langle \dot{\phi} \rangle^2) g_{ij} \end{array} \right]. \quad (40)$$

Thus

$$\Delta(\rho + 3p)^{\text{Buchert}} \rightarrow 4 \left( \langle (\dot{\phi})^2 \rangle - \langle \dot{\phi} \rangle^2 \right) \geq 0. \quad (41)$$

This agrees with the SEC calculation. The “effective  $w$  parameter” is

$$w = \frac{\langle(\dot{\phi})^2\rangle - \langle(\phi')^2\rangle - \langle\dot{\phi}\rangle^2}{\langle(\dot{\phi})^2\rangle + 3\langle(\phi')^2\rangle - \langle\dot{\phi}\rangle^2} \quad (42)$$

This contribution to stress-energy always has  $w \in (-1/3, +1)$ .

## 8. Discussion

When studying Buchert coarse-graining, working with CFLRW spacetimes is a great technical help. Specifically, a FLRW background provides you with a very nicely controlled version of Buchert coarse-graining, and CFLRW coarse-graining satisfies some nice differential identities. Indeed CFLRW coarse-graining yields an explicit and tractable effective stress-energy. The effective stress energy is not necessarily traceless. Furthermore for this form of coarse-graining the classical energy conditions are satisfied by the effective stress-energy. Finally I emphasize that Buchert coarse-graining  $\neq$  ensemble average; several key properties are radically different.

## References

1. S. R. Green and R. M. Wald, *Phys. Rev.* **D83** (2011) 084020.
2. S. R. Green and R. M. Wald, *Phys. Rev.* **D87** (2013) 12, 124037.
3. S. R. Green and R. M. Wald, *Class. Quant. Grav.* **31** (2014) 234003.
4. S. R. Green and R. M. Wald, arXiv:1506.06452 [gr-qc].
5. T. Buchert and S. Räsänen, *Ann. Rev. Nucl. Part. Sci.* **62** (2012) 57.
6. G. F. R. Ellis, *Class. Quant. Grav.* **28** (2011) 164001.
7. T. Buchert, *Class. Quant. Grav.* **28** (2011) 164007, [arXiv:1103.2016 [gr-qc]].
8. D. L. Wiltshire, *Class. Quant. Grav.* **28** (2011) 164006.
9. T. Buchert, M. Carfora, G.F.R. Ellis, E.W. Kolb, M.A.H. MacCallum, J.J. Ostrowski, S. Räsänen, B.F. Roukema, L. Andersson, A.A. Coley, and D.L. Wiltshire, *Class. Quant. Grav.* **32** (2015) 215021.
10. T. Buchert, A. A. Coley, H. Kleinert, B. F. Roukema, and D. L. Wiltshire, arXiv:1512.03313 [astro-ph.CO].
11. J. J. Ostrowski and B. F. Roukema, arXiv:1512.02947 [gr-qc].
12. M. Visser, *Gen. Rel. Grav.* **37** (2005) 1541, [gr-qc/0411131].
13. M. Visser, *Class. Quant. Grav.* **32** (2015) 13, 135007.
14. M. Visser, in preparation.
15. M. Visser, *Science* **276** (1997) 88, [arXiv:1501.01619 [gr-qc]].
16. M. Visser, *Phys. Rev.* **D56** (1997) 7578, [gr-qc/9705070].
17. M. Visser, gr-qc/9710010.
18. M. Visser and C. Barceló, doi:10.1142/9789812792129\_0014 gr-qc/0001099.
19. C. Barceló and M. Visser, *Int. J. Mod. Phys.* **D11** (2002) 1553, [gr-qc/0205066].
20. M. Visser, *Phys. Lett.* **B349** (1995) 443 [gr-qc/9409043].